

Invariant geometry of the ideal gas

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Abstract

We analyze a Legendre invariant metric structure in the space of thermodynamic equilibrium states of an ideal gas. Due to the lack of thermodynamic interaction, the geometry turns out to be flat. We introduce the concept of thermodynamic geodesics, which correspond to quasi-static processes, analyze their properties, and show that they can be used to determine the “arrow of time” and to split the equilibrium space of the ideal gas into two completely different regions, separated by adiabatic geodesics which correspond to reversible thermodynamic processes.

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I. INTRODUCTION

The idea of using differential geometry in thermodynamics is due to Gibbs [1] who realized that the first law of thermodynamics can be represented in terms of what today is known as differential forms. Charatheodory [2] interpreted the laws of thermodynamics in an axiomatic way and in terms of Pfaffian forms. Later on, Hermann [3] introduced the concept of thermodynamic phase space where the thermodynamic variables play the role of coordinates. Additionally, Weinhold [4] and Ruppeiner [5, 6] introduced metric structures in the space of equilibrium states. The connection between the structure of the phase space and the equilibrium space was recently incorporated [7] in the formalism of *geometrothermodynamics* (GTD) in an invariant manner. In fact, a problem of using Weinhold's or Ruppeiner's metrics in equilibrium space is that the results can depend on the choice of thermodynamic potential, i. e., the results are not invariant with respect to Legendre transformations. GTD incorporates Legendre invariance into the geometric structures of the phase space and equilibrium space so that the results do not depend on the choice of thermodynamic potential or representation, a property which characterizes ordinary thermodynamics.

One of the main results of GTD is that for any thermodynamic system it delivers a Legendre invariant metric $g = g_{ab}dx^a dx^b$ for the space of equilibrium states \mathcal{E} . Here x^a are the coordinates in \mathcal{E} and the components of the metric are in general functions of the coordinates, i. e. $g_{ab} = g_{ab}(x^c)$. In this manner, the equilibrium space becomes a Riemannian manifold whose geometric properties should be related to the properties of the corresponding thermodynamic system. The most important geometric objects of a Riemannian manifold are the Christoffel symbols (or connection components)

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad} \left(\frac{\partial g_{db}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d} \right), \quad (1)$$

and the curvature tensor

$$R^a_{bcd} = \frac{\partial \Gamma^a_{bd}}{\partial x^c} - \frac{\partial \Gamma^a_{bc}}{\partial x^d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}, \quad (2)$$

which will be used in this work to investigate the properties of the equilibrium space. We assume the convention of summation over repeated indices. Furthermore, from the curvature tensor one can define the Ricci tensor $R_{ab} = g^{cd}R_{acbd}$ and the curvature scalar $R = g^{ab}R_{ab}$. In GTD, the curvature tensor is expected to be a measure of the thermodynamic interaction.

In fact, we will present a metric whose curvature tensor vanishes in the case of an ideal gas, where thermodynamic interaction is absent. The curvature tensor is not zero for systems with intrinsic thermodynamic interaction. Because of this property, the curvature of the space of equilibrium is called *thermodynamic curvature*.

The Christoffel symbols are important in connection with geodesics which are extremal curves $x^a(\tau)$ satisfying the differential equations

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0 , \quad (3)$$

where τ is an affine parameter along the geodesics. We introduce the concept of *thermodynamic geodesic* as those solutions of the geodesic equations which fulfill the laws of thermodynamics. We show that thermodynamic geodesics describe quasi-static processes and that τ can be used as a “time” parameter. Moreover, thermodynamic geodesics have a definite direction that can be interpreted as the “arrow of time”.

The main goal of this work is to present a Legendre invariant metric for the space of equilibrium states of an ideal gas and investigate in detail its geodesics. Related investigations were carried out in [8, 9, 10] by using, however, metric structures which depend on the choice of thermodynamic potential. On the contrary, we show explicitly that our results are independent of the choice of thermodynamic potential and, therefore, represent an invariant representation of the geometry of the ideal gas.

This paper is organized as follows. In Section II we present the fundamentals of GTD in the case of systems with two thermodynamic degrees of freedom. In particular, we present a Legendre invariant metric for the phase space from which it is possible to derive an invariant geometry for the space of equilibrium states. In Section III we focus on the geometry of the ideal gas and show that the corresponding thermodynamic curvature vanishes as a result of the lack of thermodynamic interaction. Furthermore, in Sections IV and V we perform a detailed analysis of the geodesic equations. The equilibrium space presents a very rich and unexpected structure that resembles the causality structure of spacetime in relativistic physics. In Appendix A we present some details of Legendre transformations and different metrics that arise in the entropy representation of the ideal gas. Finally, Section VI is devoted to discussions of our results and suggestions for further research.

II. GEOMETROTHERMODYNAMICS OF SIMPLE SYSTEMS

The simplest thermodynamic systems are those with only two thermodynamic degrees of freedom. To describe this type of systems, it is necessary to specify two extensive variables, say entropy S and volume V , two intensive variables, say temperature T and pressure P , and a thermodynamic potential, say the internal energy U . In GTD [7, 11], the thermodynamic variables are used as coordinates for the construction of the thermodynamic phase space \mathcal{T} , which in this case is 5-dimensional. We assume that \mathcal{T} is a differential manifold so that it allows the introduction of additional geometric structures. In particular, we introduce the so called fundamental Gibbs form

$$\Theta = dU - TdS + PdV , \quad (4)$$

which, as we will see below, contains the information about the first law of thermodynamics. In addition, we introduce a metric G that, in general, can depend on all the coordinates of \mathcal{T} , i.e. $G = G(U, S, V, T, P)$.

The triplet (\mathcal{T}, Θ, G) is called a Riemannian contact manifold and represents an auxiliary structure which is necessary to implement in a consistent manner the properties of ordinary thermodynamics. In particular, it is known that the thermodynamic properties of a system do not depend on the thermodynamic potential used for its description [12]. Since different thermodynamic potentials are related by means of Legendre transformations, we demand that the geometric structure of the phase space \mathcal{T} be Legendre invariant. In the case of a system with two degrees of freedom, a Legendre transformation is defined in \mathcal{T} as the change of coordinates $(U, S, V, T, P) \longrightarrow (\tilde{U}, \tilde{S}, \tilde{V}, \tilde{T}, \tilde{P})$ with the following possibilities [13]:

$$\tilde{U}_1 = U - TS , \quad S = -\tilde{T} , \quad T = \tilde{S} , \quad V = \tilde{V} , \quad P = \tilde{P} , \quad (5)$$

$$\tilde{U}_2 = U + PV , \quad S = \tilde{S} , \quad T = \tilde{T} , \quad V = \tilde{P} , \quad -P = \tilde{V} , \quad (6)$$

$$\tilde{U}_3 = U - TS + PV , \quad S = -\tilde{T} , \quad T = \tilde{S} , \quad V = \tilde{P} , \quad -P = \tilde{V} . \quad (7)$$

Usually, $\tilde{U}_1 = F$ is called the Helmholtz free energy, $\tilde{U}_2 = H$ is the enthalpy, and $\tilde{U}_3 = G$ is the Gibbs potential. If we denote by $\tilde{\Theta}_i$, $i = 1, 2, 3$, the result of applying any of the particular Legendre transformations (5)-(7) to the fundamental Gibbs form, then it is easy to see that $\tilde{\Theta}_i = d\tilde{U}_i - \tilde{T}d\tilde{S} + \tilde{P}d\tilde{V}$, showing that in fact Θ is a Legendre invariant geometric object.

In GTD we also demand that the metric G be Legendre invariant. It is possible to write down and solve the algebraic conditions that an arbitrary metric G must satisfy in order to be Legendre invariant [7]. A particular solution was found in [14] that can be written as

$$G = (dU - TdS + PdV)^2 + \Lambda \left[(ST)^{2k+1} dSdT + (VP)^{2k+1} dVdP \right] , \quad (8)$$

where k is an integer, positive or negative, and $\Lambda = \Lambda(U, S, V, T, P)$ must be chosen as a Legendre invariant function of all its arguments. As in the case of the fundamental Gibbs form, if we denote by \tilde{G}_i the result of applying any of the Legendre transformations (5)-(7) to the metric (8), the Legendre invariance of G becomes clear. In fact, the functional dependence of \tilde{G}_i coincides with (8) with U replaced by \tilde{U}_i , S by \tilde{S} , and so on. In this manner, we see that the particular triplet (\mathcal{T}, Θ, G) , with Θ and G given as in Eqs.(4) and (8), respectively, is invariant with respect to all possible Legendre transformations in the case of systems with two thermodynamic degrees of freedom. This is an important property which guarantees that our further results are independent of the choice of thermodynamic potential.

The next important element of GTD is the space of equilibrium states \mathcal{E} which, in quite general terms, is the space where systems in thermodynamic equilibrium can exist and their properties can be investigated. This means that \mathcal{E} is a 2-dimensional subspace of \mathcal{T} which we define in the following manner. Let us choose the set of extensive variables (S, V) as the coordinates of \mathcal{E} . Then, when evaluated on \mathcal{E} , the remaining coordinates of \mathcal{T} must be functions of U and V only, i. e.

$$U = U(S, V) , \quad T = T(S, V) , \quad P = P(S, V) . \quad (9)$$

The first of these equations is known as the fundamental equation in the energy representation. In fact, once $U(S, V)$ is given explicitly, one can derive all the equations of state and thermodynamic properties of the corresponding thermodynamic system. To guarantee the existence of the second and third equations of (9), we demand that the projection of the fundamental Gibbs form on \mathcal{E} vanishes, i. e.,

$$\Theta|_{\mathcal{E}} = 0 \iff dU = TdS - PdV , \quad (10)$$

a relationship that is immediately recognized as the first law of thermodynamics. Furthermore, since $U = U(S, V)$, the first law of thermodynamics implies that

$$\frac{\partial U}{\partial S} = T , \quad \frac{\partial U}{\partial V} = -P , \quad (11)$$

so that T and P become functions of S and V , as stated in (9). In ordinary thermodynamics, the relationships (11) represent the conditions for thermodynamic equilibrium. As for the metric G of \mathcal{T} , we demand that its projection on \mathcal{E} , by using (10) and (11), induces a metric g on \mathcal{E} , i. e., $G|_{\mathcal{E}} = g = g(S, V)$. In the particular case of the Legendre invariant metric (8), a straightforward calculation leads to

$$g = \Lambda \left\{ \left(S \frac{\partial U}{\partial S} \right)^{2k+1} \frac{\partial^2 U}{\partial S^2} dS^2 + \left(V \frac{\partial U}{\partial V} \right)^{2k+1} \frac{\partial^2 U}{\partial V^2} dV^2 \right. \quad (12)$$

$$\left. + \left[\left(S \frac{\partial U}{\partial S} \right)^{2k+1} + \left(V \frac{\partial U}{\partial V} \right)^{2k+1} \right] \frac{\partial^2 U}{\partial S \partial V} dS dV \right\}. \quad (13)$$

If we fix the function Λ and the constant k , and specify a fundamental equation $U = U(S, V)$, the above metric is unique. This means that for a given thermodynamic system, the metric (12) determines a family of geometries that characterizes the corresponding space of equilibrium \mathcal{E} . In fact, we will see that the freedom contained in the choice of Λ and k can be used to impose further physical conditions on the geometry of \mathcal{E} .

The above description corresponds to the energy representation in ordinary thermodynamics. One of the advantages of GTD is that the formalism allows us to handle different representations in an invariant way. For later purposes, we present here the entropy representation for a system with two thermodynamic degrees of freedom. A simple rearrangement of Eq.(4) leads to the Gibbs form in the entropy representation

$$\Theta_S = dS - \frac{1}{T} dU - \frac{P}{T} dV, \quad (14)$$

so that the coordinates of the phase space \mathcal{T} are $(S, U, V, 1/T, P/T)$, and the metric G corresponding to (8) can be written as

$$G_S = \left(dS - \frac{1}{T} dU - \frac{P}{T} dV \right)^2 + \Lambda \left[\left(\frac{U}{T} \right)^{2k+1} dU d\left(\frac{1}{T} \right) + \left(\frac{VP}{T} \right)^{2k+1} dV d\left(\frac{P}{T} \right) \right]. \quad (15)$$

With the corresponding change of coordinates (see Appendix A), it is easy to show that the above geometric objects are invariant with respect to Legendre transformations. Furthermore, for the equilibrium subspace \mathcal{E} we choose the extensive variables U and V so that the remaining coordinates become functions of U and V when projected on \mathcal{E} . In particular, the fundamental equation must now be given as $S = S(U, V)$. As before, we demand that the projected Gibbs form and metric satisfy

$$\Theta_S|_{\mathcal{E}} = 0, \quad G_S|_{\mathcal{E}} = g_S = g_S(S, V), \quad (16)$$

so that from the first condition we obtain the first law of thermodynamics and the conditions for thermodynamic equilibrium in the entropy representation:

$$dS = \frac{1}{T}dU + \frac{P}{T}dV, \quad \frac{\partial S}{\partial U} = \frac{1}{T}, \quad \frac{\partial S}{\partial V} = \frac{P}{T}. \quad (17)$$

Moreover, the metric g_S of the equilibrium space can be calculated in a straightforward manner from the above equations and we obtain

$$g_S = \Lambda \left\{ \left(U \frac{\partial S}{\partial U} \right)^{2k+1} \frac{\partial^2 S}{\partial U^2} dU^2 + \left(V \frac{\partial S}{\partial V} \right)^{2k+1} \frac{\partial^2 S}{\partial V^2} dV^2 + \left[\left(U \frac{\partial S}{\partial U} \right)^{2k+1} + \left(V \frac{\partial S}{\partial V} \right)^{2k+1} \right] \frac{\partial^2 S}{\partial U \partial V} dU dV \right\}. \quad (18)$$

Again we see that once the fundamental equation $S = S(U, V)$ is given, the metric of \mathcal{E} is uniquely determined, up to a constant k and the conformal function Λ . As in ordinary thermodynamics, the fundamental equation must satisfy the second law which in the case under consideration can be written as [12]

$$\frac{\partial^2 U}{\partial E^a \partial E^b} \geq 0, \quad \frac{\partial^2 S}{\partial F^a \partial F^b} \leq 0, \quad (19)$$

for the energy and entropy representation, respectively. Here $a, b = 1, 2$, $E^a = (S, V)$ and $F^a = (U, V)$.

It is worth mentioning that in the above construction we chose the extensive variables as coordinates for the equilibrium space \mathcal{E} in order to obtain the energy and entropy representation which are the most common approaches used in ordinary thermodynamics. However, in general it is possible to choose any 2-dimensional subspace of \mathcal{T} to define the equilibrium space \mathcal{E} which would correspond to a different representation in ordinary thermodynamics. The Legendre invariance of the phase space \mathcal{T} has as a consequence that all possible representations are equivalent and the properties of thermodynamic systems do not depend on the representation.

The above geometric construction can be applied to any thermodynamic system with two degrees of freedom. One only needs to specify the fundamental equation of the thermodynamic system in order to investigate its geometric properties. However, GTD allows a generalization to include any system with a finite number of degrees of freedom, say n . In this case, the phase space has the dimension $2n + 1$ and the subspace of equilibrium states is n -dimensional. The fundamental Gibbs form and the metrics G and g can be generalized to the $(2n + 1)$ -dimensional case in a straightforward manner [14].

III. THE EQUILIBRIUM SPACE OF THE IDEAL GAS

In the specific case of an ideal gas, the fundamental equation in the entropy representation can be expressed as [12]

$$S(U, V) = S_0 + Nk_B c_V \ln \left(\frac{U}{U_0} \right) + Nk_B \ln \left(\frac{V}{V_0} \right) , \quad (20)$$

where c_V is the dimensionless heat capacity at constant volume, k_B is Boltzmann's constant, N is the number of particles and S_0 , U_0 and V_0 are constants. The intensive thermodynamic variables can be calculated by using the conditions of thermodynamic equilibrium (17). We obtain

$$\frac{1}{T} = \frac{Nk_B c_V}{U} , \quad \frac{P}{T} = \frac{Nk_B}{V} . \quad (21)$$

Furthermore, from Eq.(18) we obtain the simple metric

$$g_S = -(Nk_B)^{2k+2} \Lambda(U, V) \left[c_V^{2k+2} \frac{dU^2}{U^2} + \frac{dV^2}{V^2} \right] . \quad (22)$$

The arbitrariness contained in the function Λ and the constant k can be fixed by demanding that g_S determines an extremal surface [15]. This condition is very common in physics and is usually interpreted as a consequence of the principle of minimum action. In fact, the metric (22) determines the infinitesimal area element $dA = \sqrt{|\det(g_S)|} dU dV$ so that the calculation of an arbitrary area in the space determined by g_S implies the calculation of the integral $A = \int \sqrt{|\det(g_S)|} dU dV$. If we demand that the area be an extremal, i. e. $\delta A = 0$, where δ represents the variation, we obtain a set of differential equations [14], implying conditions on the form of the metric g_S . In the case of the metric (22) it can be shown that those differential equations are identically satisfied if we choose the particular solution $k = -1$ and $\Lambda = \text{const}$. For the sake of simplicity, let us choose $\Lambda = -1$ so that the metric (22) becomes

$$g_S = \frac{dU^2}{U^2} + \frac{dV^2}{V^2} \quad (23)$$

From this metric one can immediately calculate the corresponding connection and curvature, according to the formulas given in Section I, and show that the connection has non-vanishing components, but all the components of the thermodynamic curvature are zero. In GTD we interpret the absence of curvature as a manifestation of the fact that the ideal gas is characterized by the absence of thermodynamic interaction. One of the goals of GTD is to interpret the curvature of the space of equilibrium states as a measure of thermodynamic

interaction. This goal has been reached in the case of the ideal gas. Moreover, one can show [7, 16] that in the case of the van der Waals gas the curvature tensor of the metric (18) does not vanish. Since the van der Waals gas corresponds to a system with non-vanishing thermodynamic interaction, we interpret this result as a further indication that thermodynamic curvature can be used to measure thermodynamic interaction. In fact, this result has been proved also in the case of more exotic thermodynamic systems like black holes [17, 18, 19].

The metric for the equilibrium space of the ideal gas (23) is invariant with respect to the change of coordinates $U \rightarrow \lambda U$ and $V \rightarrow \lambda V$, where λ is an arbitrary constant. This seems to be a trivial symmetry related to a rescaling of coordinates. This is true from a mathematical point of view, but physically it implies that different thermodynamic systems can share the properties of the ideal gas. Indeed, this fact is known from statistical physics. There are (at least) three different systems which are thermodynamically equivalent: The Maxwell-Boltzmann gas, the Fermi-Dirac gas and the Bose-Einstein gas. Taking the corresponding partition functions and calculating the relevant variables one can immediately see that they are related by a rescaling of the variables U and V [20].

IV. GEODESICS IN THE EQUILIBRIUM SPACE OF THE IDEAL GAS

Since the curvature of the metric (23) vanishes, there must exist coordinates in which the metric takes the simple Euclidean form. In fact,

$$g_S = d\xi^2 + d\eta^2, \quad \xi = \ln U, \quad \eta = \ln V, \quad (24)$$

where for simplicity we set the additive constants of integration such that $\xi \geq 0$ and $\eta \geq 0$. This form of the metric is specially appropriate for the investigation of geodesics. As we mentioned in Section I, geodesics of a space represents curves of extremal length. In the case of the equilibrium space \mathcal{E} we can define the thermodynamic length as [14]: $L = \int \sqrt{g_S}$. Then, the condition of extremal thermodynamic length, $\delta L = 0$, leads to the geodesic equations for the coordinates of the space \mathcal{E} . In the case of the flat metric (24), all the components of the connection vanish and the geodesic equations become $\ddot{\xi} = 0$ and $\ddot{\eta} = 0$, where the dot denotes differentiation with respect to the affine parameter τ . The solutions are found to be $\xi = \xi_1 \tau + \xi_0$ and $\eta = \eta_1 \tau + \eta_0$, where ξ_0 , ξ_1 , η_0 and η_1 are constants, i. e.,

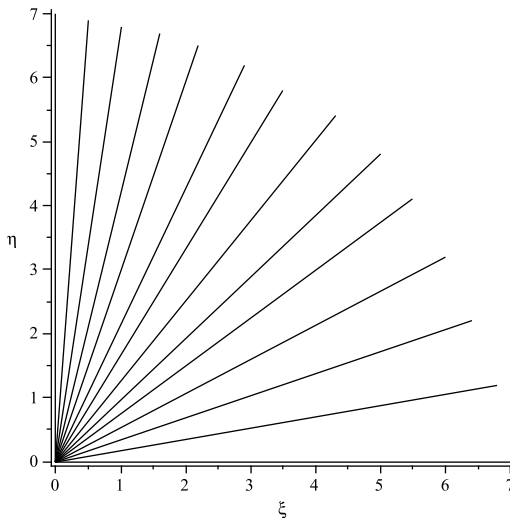


FIG. 1: Geodesics in the space of equilibrium states of the ideal gas in logarithmic coordinates $\xi = \ln U$ and $\eta = \ln V$. The initial state is located on the origin $(0,0)$. In general, two arbitrary equilibrium states can always be connected by means of a geodesic.

they represent straight lines. For instance, consider all geodesics with initial state $\xi_i = 0$ and $\eta_i = 0$. Then, on the $\xi\eta$ -plane the geodesics must be contained within the quadrant determined by $\xi \geq 0$ and $\eta \geq 0$, due to our choice of integration constants for ξ and η that also fixes the values of the constants ξ_0 , ξ_1 , η_0 and η_1 . Consequently, the geodesics of the ideal gas can be depicted by using the equation $\xi = c_1\eta + c_0$, with constants c_0 and c_1 . For any arbitrary initial state, there is always a straight line that connects that state with any arbitrary point on the $\xi\eta$ -plane. This means that the entire space of equilibrium states can be covered by geodesics. This behavior is schematically illustrated in figure 1.

However, not all the solutions of the geodesic equations must be physically realistic. Indeed, there could be straight lines connecting equilibrium states that are not compatible with the laws of thermodynamics. In particular, one would expect that the second law of thermodynamics imposes strong requirements on the solutions. In ordinary thermodynamics two equilibrium states are related to each other only if they can be connected by means of quasi-static process. Then, a geodesic that connects two physically meaningful states can be interpreted as representing a quasi-static process. Since a geodesic curve is a dense succession of points, we conclude that a quasi-static process can be seen as a dense succession of equilibrium states, a statement which coincides with the definition of quasi-static processes

in equilibrium thermodynamics [12]. Furthermore, the affine parameter τ can be used to label all equilibrium states which belong to a geodesic. Since the affine parameter is defined up to a linear transformation, it should be possible to choose it in such a way that it increases as the entropy of a quasi-static process increases. This opens the possibility of interpreting the affine parameter as a “time” parameter with a specific direction which coincides with the direction of entropy increase.

In the special case of the ideal gas, the fundamental equation (20) in coordinates ξ and η represents a straight line

$$S = S_0 + Nk_B c_V \xi + Nk_B \eta , \quad (25)$$

so that the entropy increases as ξ and η increases. Consequently, any straight line pointing outwards of the initial zero point and contained inside the allowed positive quadrant connect states with increasing entropy. This behavior is schematically illustrated in figure 2 where the arrows indicate the direction in which a quasi-static process can take place, i.e., in which the entropy increases. A quasi-static process connecting states in the opposite direction is not allowed by the second law of thermodynamics. Consequently, the affine parameter τ represents a time parameter and the direction on each geodesic indicates the “arrow of time”.

If the initial state is not at the origin of the $\xi\eta$ -plane, the second law permits the existence of geodesics for which one of the coordinates, say η , decreases as long as the other coordinate ξ increases in such a way that the entropy increases or remains constant. In fact, the region in the $\xi\eta$ -plane available from a given initial equilibrium state is situated within two extreme geodesics which span a maximum angle that can be determined in the following way. Let the initial state be at the point (ξ_i, η_i) . According to the second law of thermodynamics and Eq.(25), the state characterized by the coordinate values (ξ_f, η_f) can be reached by a geodesic with origin at (ξ_i, η_i) if the condition

$$c_V \Delta\xi + \Delta\eta \geq 0 , \quad \Delta\xi = \xi_f - \xi_i , \quad \Delta\eta = \eta_f - \eta_i , \quad (26)$$

is satisfied. Consider geodesics for which $\Delta\xi < 0$. Hence, only those geodesics are allowed for which $\Delta\eta \geq c_V |\Delta\xi|$. The equal sign determines the extreme geodesic with constant entropy which intersects the η -axis at the point $\xi_f = 0$ and $\eta_f = \eta_i + c_V |\Delta\xi| = \eta_i + c_V \xi_i$. This geodesic intersects the η -axis at an angle α such that $\tan \alpha = 1/c_V$. Consider now geodesics with $\Delta\eta < 0$. The allowed geodesics must satisfy $\Delta\xi \geq |\Delta\eta|/c_V$ and the extreme

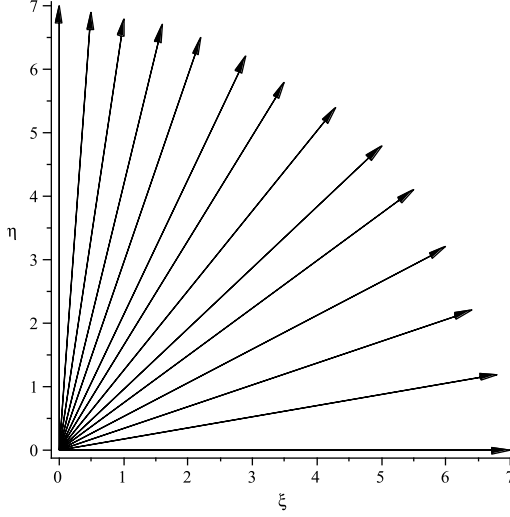


FIG. 2: Geodesics that satisfy the second law of thermodynamics. The initial equilibrium state is located on the origin of coordinates. There is only one geodesic which connects the origin with any other equilibrium state. The arrows show the direction in which entropy increases, suggesting that they could be interpreted as the “arrows of time”.

adiabatic geodesic intersects the ξ -axis at the point with coordinates $\eta_f = 0$ and $\xi_f = \xi_i + |\Delta\eta|/c_v = \xi_i + \eta_i/c_v$. The adiabatic geodesic intersects the ξ -axis at an angle α' with $\tan \alpha' = c_v$. Since the intersection angles are complementary, $\tan \alpha' = 1/\tan \alpha$, we conclude that the angle spanned by the two adiabatic geodesics (one with $\Delta\xi < 0$ and the second one with $\Delta\eta < 0$) is π .

An alternative derivation of the above geometric construction of adiabatic geodesics consists in considering the corresponding equation in the form $c_v(\xi_f - \xi_i) + \eta_f - \eta_i = 0$, which can be rewritten as

$$\frac{\xi_f}{\xi_i + \eta_i/c_v} + \frac{\eta_f}{\eta_i + \xi_i c_v} = 1, \quad (27)$$

and is immediately recognized as the equation of a straight line. This line in the equilibrium space can be occupied only by states belonging to an adiabatic process. Moreover, since the entropy remains constant along this straight line, the “arrow of time” can point in both directions. This is illustrated in figure 3. From the last equation it is then easy to obtain the general relationships

$$\tan \alpha = \frac{\xi_i + \eta_i/c_v}{\eta_i + \xi_i c_v}, \quad \tan \alpha' = \frac{\eta_i + \xi_i c_v}{\xi_i + \eta_i/c_v}. \quad (28)$$

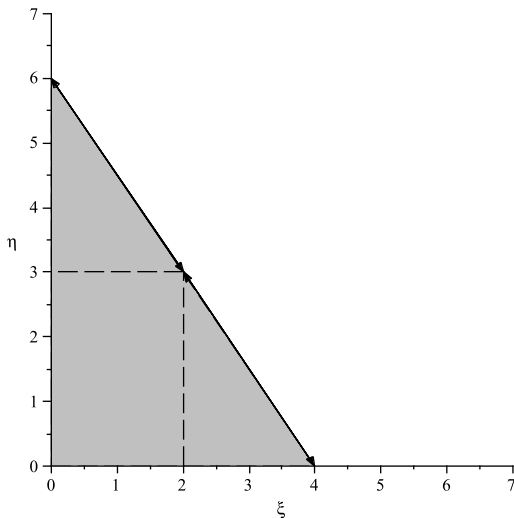


FIG. 3: Adiabatic geodesics with initial state at $\eta_i = 3$, $\xi_i = 2$. They represent reversible processes so that the “arrow of time” can point in both directions. States in the shadow region are connected to the initial state by geodesics with a negative change in the entropy.

These formulas are valid for any values of the initial state, except the one situated on the origin of coordinates. In fact, for the initial states $(0, \eta_i)$ and $(\xi_i, 0)$, we recover the values of the intersection angles described above. If the initial state is on the origin, the entropy condition (26) is always satisfied since any arbitrary straight line that starts at the origin is characterized by $\Delta\xi \geq 0$ and $\Delta\eta \geq 0$ so that the allowed geodesics could occupy the entire positive quadrant as illustrated in figure 2. However, this result changes drastically if we take into account the third law of thermodynamics which postulates the impossibility of reaching absolute zero of temperature or, equivalently, the minimum value of the entropy. For the ideal gas with fundamental equation (25) the minimum value for the entropy is S_0 and corresponds to $\xi = 0$ and $\eta = 0$. Consequently, the origin of coordinates must be “removed” from the space of equilibrium states.

The above results suggest the introduction of the concept of *thermodynamic geodesics* as those solutions of the geodesic equations which satisfy the laws of thermodynamics. In the case of the ideal gas, a thermodynamic geodesic can be represented as a straight line that never crosses the origin of coordinates and possesses a definite direction which coincides with the direction of entropy increase.

We see that the laws of thermodynamics imply that the geometric structure of the equi-

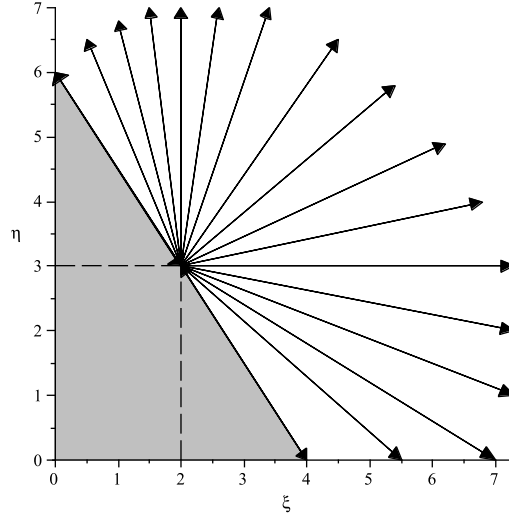


FIG. 4: General structure of the thermodynamic geodesics in the space of equilibrium states of an ideal gas. Here we choose a monoatomic gas so that $c_v = 3/2$. Consequently, $\alpha \approx 33.3^\circ$ and $\alpha' \approx 56.7^\circ$. The shadow region contains all the states that due to the second law of thermodynamics cannot be reached by thermodynamic geodesics with the fixed initial state. Adiabatic geodesics determine the boundary of the connectivity region where several thermodynamic geodesics are depicted.

librium space is as illustrated in figure 4. For any given initial equilibrium state (ξ_i, η_i) , there exist two different regions. The first one is determined by all the states than can be reached from the initial state by means of quasi-static processes, i. e., by thermodynamic geodesics. This could be called the *region of connectivity* of the initial state. If we identify τ as a time parameter, the connectivity region acquires the characteristics of a causally-connected region, resembling concepts of relativistic physics. The second region is composed of all the equilibrium states that cannot be reached from the initial state by thermodynamic geodesics. We call it the *region of non-connectivity*. Again, it could be also identified with the non-causally connected region of spacetime in relativistic physics. The boundary between the connectivity and non-connectivity regions is occupied by adiabatic thermodynamic geodesics and this the only place in the equilibrium space where reversible thermodynamic processes can occur.

It is interesting to mention that the area of the connectivity region is infinite, whereas the non-connectivity region has a finite area. Indeed, from the above geometric construction

(see figure 3) we see that the non-connectivity region of an arbitrary initial state (ξ_i, η_i) corresponds to a triangle of area

$$A_{NC} = \frac{1}{2c_V} (c_V \xi_i + \eta_i)^2 . \quad (29)$$

Since the heat capacity is related to the number of degrees of freedom per particle, say l , as $c_V = l/2$ (see, for instance, [20]), we see that for ordinary ideal gases with finite l the area of the non-connectivity region remains finite and increases as l grows. If we extrapolate this result to field theories, where the number of degrees of freedom tends to infinity, the non-connectivity region tends to occupy the entire space of equilibrium states. This result seems to be interesting in the context of thermodynamics of black holes.

V. GEODESICS IN TERMS OF THERMODYNAMIC VARIABLES

Since one of the most important properties of GTD is its invariance, we can carry out analysis in arbitrary coordinates. For the sake of generality, we will present in this section the analysis of the geodesic equations of the space of equilibrium space of the ideal gas in more conventional thermodynamic coordinates. Consider the flat metric (23). According to the formula given in Section I, the Christoffel symbols for this metric can be written as

$$\Gamma_{UU}^U = -\frac{1}{U} , \quad \Gamma_{VV}^V = -\frac{1}{V} , \quad (30)$$

and the remaining components vanish identically. Then, the geodesic equations (3) become

$$\frac{d^2 U}{d\tau^2} - \frac{1}{U} \left(\frac{dU}{d\tau} \right)^2 = 0 , \quad \frac{d^2 V}{d\tau^2} - \frac{1}{V} \left(\frac{dV}{d\tau} \right)^2 = 0 , \quad (31)$$

where τ is an affine parameter along the geodesics. These equations can easily be integrated and yield $U = U_0 e^{\tau/\tau_U}$, and $V = V_0 e^{\tau/\tau_V}$, respectively, where U_0 , V_0 , τ_U and τ_V are constants. Then, from these solutions we obtain the relationship

$$\frac{U^{\tau_U}}{V^{\tau_V}} = \text{const} , \quad (32)$$

which allows us to illustrate the form of the geodesics in a simple manner. Some examples are depicted in figure 5.

The relationship (32) means that along any geodesic the ratio U^{τ_U}/V^{τ_V} is a conserved quantity. The explicit values of the constants τ_U and τ_V depend on the type of process

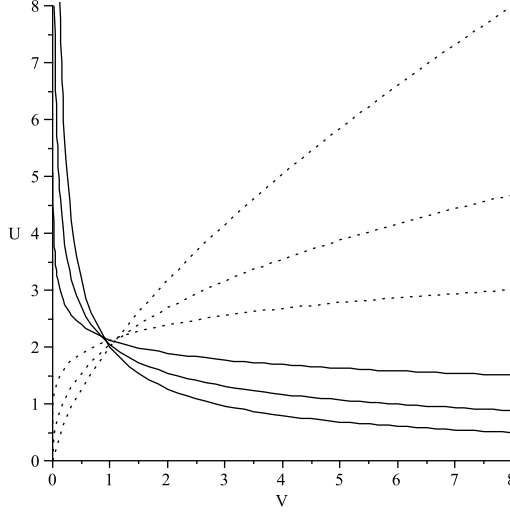


FIG. 5: Geodesics in terms of the thermodynamic variables U and V for different relaxation times τ_U and τ_V . Two different types of geodesics are plotted with $\tau_U/\tau_V > 0$ (dotted lines) and $\tau_U/\tau_V < 0$ (solid lines).

represented by the geodesic. Consider for instance an adiabatic process for which, according to the fundamental equation for the ideal gas (20), $U^{c_V} V = \text{const}$. Then, a comparison with Eq.(32) yields

$$\frac{\tau_U}{\tau_V} = -c_V . \quad (33)$$

In the case of a polytropic process with $PV^n = \text{const}$, where $n = n(c_V)$ is the polytropic coefficient, we use the state equations (21) to rewrite (32) as $(c_V P)^{\tau_U} V^{\tau_U - \tau_V} = \text{const}$ so that

$$n = 1 - \frac{\tau_V}{\tau_U} . \quad (34)$$

The relationship (32) is, however, completely general and the constants τ_U and τ_V do not necessarily need to be related to the heat capacity c_V . This is in agreement with further special cases in which the explicit values of the constants τ_U and τ_V follow from the analysis of different statistical ensembles (see, for instance, [20], §56). The fact that we obtain the parameters τ_U and τ_V as constants of integration of the geodesic equations, in which τ could be interpreted as a “time” parameter, suggests the possibility of identifying them as related to the “relaxation time” of the thermodynamic variables U and V , respectively.

Due to the Legendre invariance of GTD it is possible to perform similar analysis for any arbitrary thermodynamic potential. In Appendix A we present all possible thermodynamic

potentials and the corresponding metrics for the space of equilibrium states. There, it is shown that in all cases the solutions to the geodesic equations can be expressed as $Z^I = Z_0^I \exp(\tau/\tau_I)$, where $Z^I = (U, V, 1/T, P/T)$ and Z_0^I and τ_I are constants. Then, we have that

$$\frac{(Z^I)^{\tau_I}}{(Z^J)^{\tau_J}} = \text{const} \quad (35)$$

is a universal relationship that holds for all thermodynamic processes of an ideal gas. It means that for all thermodynamic variables there exist “relaxation times” which determine conserved quantities along thermodynamic geodesics.

It should be emphasized that we are using the term “relaxation time” in a very broad sense. We can interpret τ as a time parameter and, consequently, τ_I as a relaxation time, only in the context of quasi-static processes. This means that we must limit ourselves to the consideration of systems characterized by infinitely slow changes, measured on a intrinsic time scale, so that after any perturbation the system reaches equilibrium much faster, almost instantaneously, than its physical parameters vary.

VI. CONCLUSIONS

In this work we present a detailed analysis of the geometry of the ideal gas. The main property of our analysis is its invariance with respect to Legendre transformations, i. e., it is independent of the choice of thermodynamic potential, a property which is essential in ordinary thermodynamics. The geometry of the ideal gas turns out to be flat, in accordance with our intuitive expectation that the absence of thermodynamic interaction would imply absence of curvature. This is an indication that thermodynamic curvature can be used as a measure of thermodynamic interaction. In fact, this result has been shown to hold also for more general thermodynamic systems, like the van der Waals gas, and more exotic systems, like black holes.

Our analysis of geodesics of the space of equilibrium space shows that they can be represented as straight lines when logarithmic thermodynamic variables are used. We introduce the concept of thermodynamic geodesics as those solutions of the geodesic equations which satisfy the laws of thermodynamics. Then, the equilibrium space can be represented as a Cartesian-like plane where thermodynamic geodesics correspond to quasi-static processes. The third law of thermodynamics implies that the origin of coordinates must be removed

from the space of equilibrium states. This opens the possibility of interpreting the third law as a topological property of the equilibrium space.

For any given initial state, the equilibrium space can be split into two different regions. The connectivity region is occupied by all states which are connected to the initial one by means of thermodynamic geodesics. On the contrary, the region of non-connectivity corresponds to those equilibrium states than cannot be reached from the initial state by using only thermodynamic geodesics. In the boundary between these two regions there exist adiabatic thermodynamic geodesics. It can be shown that this is the only place where adiabatic geodesics can exist so that the boundary determines the only region in the equilibrium space where reversible thermodynamic processes can take place.

In this work we used the entropy representation. For the sake of completeness, we performed the same analysis using the energy representation. The main difference in the analysis follows from the fact that the fundamental equation (20) in the energy representation, $U(S, V) = U_0 \exp(S/Nk_B c_V)/V^{c_V}$, leads to a non-diagonal term in the metric of the space of equilibrium states. This complicates the analysis of the geodesic equations. Nevertheless, the corresponding metrics with all possible thermodynamic potentials were shown to describe a flat space so that in all the cases it is possible to introduce Cartesian-like coordinates and the investigation of the geodesic equations leads to results equivalent to those obtained in the entropy representation. This result corroborates one of the most important properties of the formalism of GTD, i. e., the choice of thermodynamic potential or representation does not affect the results.

We found that the space of equilibrium states of the ideal gas possesses a very rich geometric structure which resembles the structure of spacetime in relativistic physics. Since this structure is the result of applying the laws of thermodynamics in a geometric context, we expect similar structures in the case of more general thermodynamic systems. For instance, the metric for the equilibrium space of the van der Waals gas can be obtained immediately from Eq.(18) by introducing the corresponding fundamental equation $S = S(U, V)$. The resulting metric, however, is no longer flat. This is due to the fact that the van der Waals gas possesses a certain degree of thermodynamic interaction which, according to the formalism of GTD, generates a non-vanishing thermodynamic curvature. As a result, the geodesic equations are much more complicated and there are no Cartesian-like coordinates in which the geodesics could be represented as straight lines. Preliminary calculations indicate that

in this case the geodesic equations must be analyzed by using numerical methods. Due to the importance of the van der Waals gas, we expect to attack this problem in the future.

In this work we limit ourselves to systems with only two thermodynamic degrees of freedom. We can increase the number of degrees of freedom, maintaining the flatness of the equilibrium space, by considering, for instance, a multicomponent ideal gas, a paramagnetic ideal gas, etc. In all these case the flatness of the equilibrium space implies that there exist Cartesian-like coordinates. The analysis of thermodynamic geodesics would then require to consider the topological properties and geometric properties of equilibrium spaces in higher dimensions.

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APPENDIX A: THE ENTROPY REPRESENTATION

As mentioned in Section III, in the entropy representation the coordinates of the phase space \mathcal{T} are $(S, U, V, \beta, \vartheta)$, with $\beta = 1/T$ and $\vartheta = P/T$ so that the auxiliary metric in \mathcal{T} can be written as

$$G_S = (dS - \beta dU - \vartheta dV)^2 + \Lambda \left[(U\beta)^{2k+1} dU d\beta + (V\vartheta)^{2k+1} dV d\vartheta \right] . \quad (\text{A1})$$

As before, the structure of this auxiliary metric is such that any extensive variable is multiplied by its corresponding intensive variable. As a result the above metric is invariant with respect to the following Legendre transformations, $(S, U, V, \beta, \vartheta) \longrightarrow (\tilde{S}, \tilde{U}, \tilde{V}, \tilde{\beta}, \tilde{\vartheta})$:

$$\tilde{S}_1 = S - U\beta , \quad U = -\tilde{\beta} , \quad \beta = \tilde{U} , \quad V = \tilde{V} , \quad \vartheta = \tilde{\vartheta} , \quad (\text{A2})$$

$$\tilde{S}_2 = S - V\vartheta , \quad U = \tilde{U} , \quad \beta = \tilde{\beta} , \quad V = -\tilde{\vartheta} , \quad \vartheta = \tilde{V} , \quad (\text{A3})$$

$$\tilde{S}_3 = S - U\beta - V\vartheta , \quad U = -\tilde{\beta} , \quad \beta = \tilde{U} , \quad V = -\tilde{\vartheta} , \quad \vartheta = \tilde{V} . \quad (\text{A4})$$

The thermodynamic potentials \tilde{S}_1 , \tilde{S}_2 , and \tilde{S}_3 are known as Massieu functions [12]. For a given fundamental equation $S = S(U, V)$ they represent the same subspaces of \mathcal{T} in different coordinates. In the case of the ideal gas, the Messieu functions can be derived explicitly by

using the state equations (21) in the form $U\beta = c_V Nk_B$ and $V\vartheta = Nk_B$ and the fundamental equation (20). We get (dropping the tildes)

$$S_1(\beta, V) = S_{01} - c_V Nk_B \ln \beta + Nk_B \ln V , \quad (\text{A5})$$

$$S_2(U, \vartheta) = S_{02} + c_V Nk_B \ln U - Nk_B \ln \vartheta , \quad (\text{A6})$$

$$S_3(\beta, \vartheta) = S_{03} - c_V Nk_B \ln \beta - Nk_B \ln \vartheta , \quad (\text{A7})$$

where S_{01} , S_{02} , and S_{03} are constants. Then, the metric of the subspace of equilibrium states \mathcal{E} in each case corresponds to

$$g_1 = \frac{d\beta^2}{\beta^2} + \frac{dV^2}{V^2} , \quad (\text{A8})$$

$$g_2 = \frac{dU^2}{U^2} + \frac{d\vartheta^2}{\vartheta^2} , \quad (\text{A9})$$

$$g_3 = \frac{d\beta^2}{\beta^2} + \frac{d\vartheta^2}{\vartheta^2} , \quad (\text{A10})$$

where we have chosen $\Lambda = -1$ and $k = 1$ in the auxiliary metric (A1). Clearly, all these metrics represent the same flat space of equilibrium states for the ideal gas. The geodesic equations can be solved as explained in Section III and we obtain $Z^I = Z_0^I \exp(\tau/\tau_I)$, where $Z^I = (U, V, \beta, \vartheta)$ and Z_0^I and τ_I are constants. This is a concrete example of the invariance of the results obtained by using the formalism of GTD.

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